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Non-life insurance pricing: multi-agent model

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Abstract. We use the maximum entropy principle for the pricing of non-life insurance and recover the Bühlmann results for the economic premium principle. The concept of economic equilibrium is revised in this respect.

PACS. 89.65.Gh Economics; econophysics, financial markets, business and management – 05.20.-y Classical statistical mechanics

1 Introduction

Recently economists have shown much interest in using methods borrowed from statistical mechanics to study the financial markets and economic systems. The concept of entropy due to its increasing nature in the all thermal phenomena was the main reason for their attention. The conceptual revision of economic equilibrium is one of the applications of the maximum entropy principle in economics. The inadequacy of mechanical equilibrium (Walrasian) to describe some real market features [1] persuaded some people to change their mind about this model. A statistical description of equilibrium appears to be the most suitable way for dealing with those problems that cannot be explained by a mechanical picture of equilibrium [2–14]. Another way that economics or finance may benefit from the entropy concept is in asset pricing. Although the Black-Scholes model has had tremendous success in option pricing, it fails to work for incomplete markets. It only appears to be an approximate way for price prediction in the financial market. Minimal relative entropy is an alternative method for option pricing in this case [15–19]. The Black-Scholes model itself may be derived through application of this method [20]. The concept of maximum entropy is also a prevalent method for estimation and inference from economic data [21].

Insurance is an important part of the financial market. The study of insurance markets by the aid of statistical mechanics was begun by the author and his colleague studying the financial reaction of an insurance company to the variation in the number of insurants (policy holders) [22]. In subsequent works we also suggested a way for pricing the insurance premium on the basis of equilibrium statistical mechanics [11–14, 23, 24]. In analogy with a thermal system which is immersed in a heat reservoir and exchanges energy with it, we consider an economic system which is surrounded by the other economic agents in the market. The economic system such as an insurance company interacts with its environment, the rest of the financial market, by exchanging money. In the equilibrium state the probability for exchanging specified amount of money is similar to probability of the exchanged energy in the thermal systems.

In this paper we proceed with the same idea for using the maximum entropy principles in premium calculations but restricting ourselves to a multi-agent model for the insurance market. This model was used previously in the actuarial literature [25, 26].

2 Bühlmann economic premium principle

The insurance companies and buyers of insurance are the typical economic agents in the financial market. They compete with each other to benefit more from their trade. The utility function demonstrates the specified amounts of profit that an agent is interested in making. Common sense tells us, the agent's utility function should depend on its financial status which is frequently described by its wealth, $u(W)$. It is assumed that the utility function has a positive first derivative, $u'(W) > 0$, to guarantee that the profit is desirable for the agent, and negative second derivative, $u''(W) < 0$, to restrict its cupidity (a rational agent should be risk averse). The risk aversion parameter, $\beta(W) = -u''(W)/u'(W)$, is also involved in the utility function, to scale the agent's desire in the market with respect to its wealth.

Equilibrium is attained when all agents are satisfied with their trade. In other words their utility functions should be a maximum in the equilibrium state [27]. This condition should be expressed in an average form because of the existence of risks in the market which alter the

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agent's wealth randomly

$$\int_{\Omega} u_i(W_i(\omega))d\Pi(\omega) = max. \quad (1)$$

where ω stands for an element of the risk's probability space Ω. The measure of the integral demonstrates the weight for an occurrence of a random event (risk). Thus we have,

$$\int_{\Omega} d\Pi(\omega) = 1. \quad (2)$$

The index i in equation (1) distinguishes the different agents.

Each agent in the market will incur a loss, $X_i(\omega)$ if ω happens. He has insured himself for the amount $\langle Y_i \rangle$ and receives $Y_i(\omega)$ upon occurrence of this event. The insurance price is given by,

$$\langle Y_i \rangle = \int_{\Omega} \varphi(\omega) Y_i(\omega) d\Pi(\omega). \quad (3)$$

The function $\varphi : \Omega \to \mathbb{R}$ is called price density. The agent's wealth also varies due to this trading as follows,

$$W_i(\omega) = W_{0i} - X_i(\omega) + Y_i(\omega) - \langle Y_i \rangle. \quad (4)$$

We suppose the market is a closed system hence the clearing condition is satisfied,

$$\sum_i Y_i(\omega) = 0. \quad (5)$$

The sum is over all agents in the market. The above equation in addition to equation (1) allows us to find the price density [25,26],

$$\varphi(\omega) = \frac{e^{\beta Z(\omega)}}{\int_{\Omega} e^{\beta Z(\omega)} d\Pi(\omega)} \quad (6)$$

where $Z(\omega)$ is the aggregate loss in the market,

$$Z(\omega) = \sum_i X_i(\omega). \quad (7)$$

The coefficient β comes from a combination of risk aversion parameters of different agents,

$$\frac{1}{\beta} = \sum_i \frac{1}{\beta_i}. \quad (8)$$

Equation (6) was derived for the first time by Bühlmann in his famous articles [25,26]. In the following section we retrieve this result again based on the maximum entropy principle.

3 The maximum entropy method in economics

Risks induce random conditions in the market even if the agents have definite states at the beginning. The randomness in the market will increased as time goes forward. Eventually the market will reach a state with maximum randomness. This is what we refer to as the equilibrium state. The consequence of randomness in a market is the loss of information about the agents and their strategies of trading.

The main question for an economist is how they can calculate the probability for acquiring the specified amount of money by an agent in the market. As is seen in equation (3) the insurance price is defined with respect to this probability function, $\varphi(\omega)$, which is called the price density in actuarial terminology.

When we make inferences about an unknown distribution based on only a few restrictions, the maximum entropy principle appears to be the best way [28]. We adopt this method for calculating the above mentioned probability density.

The Shannon-Jaynes entropy functional can be written as [28],

$$H[\varphi] = -\int_{\Omega} \varphi(\omega) \ln \varphi(\omega) d\Pi(\omega). \quad (9)$$

The price density should satisfy a normalization condition,

$$\int_{\Omega} \varphi(\omega) d\Pi(\omega) = 1. \quad (10)$$

The wealth of the market is defined as the sum of the agents wealth,

$$W(\omega) = \sum_i W_i(\omega). \quad (11)$$

We assume that the average of the market's wealth is constant. This is a legal assumption for the exchange (conservative) market [2,6,25,26],

$$\langle W \rangle = \int_{\Omega} \varphi(\omega) W(\omega) d\Pi(\omega) = \text{const.} \quad (12)$$

At equilibrium, the entropy equation (9) has a maximum value and the constraints, i.e. equations (10) and (12), should also be satisfied simultaneously. This mathematical problem can be solved immediately by the method of Lagrange's multipliers,

$$\delta H[\varphi] + \lambda \delta \int_{\Omega} \varphi(\omega) d\Pi(\omega) + \beta \delta \langle W \rangle = 0. \quad (13)$$

The solution of the above equation is called the canonical distribution. We have seen this distribution before in every statistical mechanics textbook [29],

$$\varphi(\omega) = \frac{e^{-\beta W(\omega)}}{\int_{\Omega} e^{-\beta W(\omega)} d\Pi(\omega)}. \quad (14)$$

The above result is confirmed analytically [4–6], by simulation [7,10] and empirical data [8,9].

There is another approach to obtain equation (14) for the agent-environment model of financial markets [11,12,14]. This approach is also applicable here for the multi-agent model.

Consider any economic system; here we choose the insurance market, in financial interaction with its environment. The system plus environment is isolated from the rest of the world hence the total wealth of their combination is constant,

$$W(\omega) + E(\omega) = M = \text{const.} \quad (15)$$

Let $\Omega_s(W(\omega))$ denote the number of ways that the system can acquire the wealth $W(\omega)$, common sense tell us that it should be a monotonically increasing function of its argument. In the same manner the environment also has $\Omega_e(E(\omega))$ choices for trading and finally obtains $E(\omega)$ amount of money. The macroscopic state of the market is specified by two quantities $W(\omega)$, $E(\omega)$. The whole market has then $\Omega_m(W(\omega), E(\omega))$ ways for reaching this state,

$$\Omega_m(W(\omega), E(\omega)) = \Omega_s(W(\omega))\Omega_e(M - W(\omega)). \quad (16)$$

The probability for finding the market in this specified state is,

$$\varphi(\omega) = \frac{\Omega_s(W(\omega))\Omega_e(M - W(\omega))}{\int_\Omega \Omega_s(W(\omega))\Omega_e(M - W(\omega))d\Pi(\omega)}. \quad (17)$$

If the system's wealth is smaller than the wealth of the environment then Ω_s is much smaller than Ω_e. In this case we can approximate the above probability as follows [29],

$$\varphi(\omega) \propto \Omega_e(M - W(\omega)). \quad (18)$$

The system's wealth is also very small compared to the wealth of the whole market. In particular, expand the logarithm of the above equation around M,

$$\ln \varphi(\omega) \propto \ln \Omega_e(M) - \beta W(\omega) + O(W^2). \quad (19)$$

The first term is a constant number. The higher order terms in W are so small and that we can neglect them. The parameter β is defined as,

$$\beta = \left[\frac{\partial \Omega_e(x)}{\partial x}\right]_{x=M}. \quad (20)$$

By simple algebraic manipulation one can find equation (14) for the price density.

In the case of insurance the market's wealth is given as:

$$W(\omega) = \sum_i W_{0i} - \sum_i X_i(\omega) = W_0 - Z(\omega). \quad (21)$$

Since the market's initial wealth is constant then we obtain the same form for the price density as seen in equation (6). It is worth to mentioning that the total risk must be less than the market's initial wealth.

The premium that the ith agent pays for a loss, $X_i(\omega)$, is calculated with the following formula, which is obtained from equation (6),

$$p_i = \frac{\int_\Omega X_i(\omega)e^{\beta Z(\omega)}d\Pi(\omega)}{\int_\Omega e^{\beta Z(\omega)}d\Pi(\omega)}. \quad (22)$$

If the risk functions for different agents have no correlation and dependency then the Esscher principle is obtained [25,26],

$$p_i = \frac{\int_{\Omega_i} X_i(\omega)e^{\beta X_i(\omega)}d\Pi(\omega)}{\int_{\Omega_i} e^{\beta X_i(\omega)}d\Pi(\omega)}. \quad (23)$$

The Esscher principle is a successful method for pricing the risk insurance [30] and also other assets [31–34].

The parameter β plays an important role in the price density. It can be calculated on the basis of the method introduced in previous work [14,23,24]. Our intuition from similar cases in statistical mechanics tells us [29,35],

$$\beta \sim \frac{1}{\langle W \rangle}. \quad (24)$$

This result shows that the wealthier market offers lower prices and the prices also depend on the size of the risks in the market.

This adopted way for calculating the premium is more general and independent of the market's models; namely multi-agents or the agent-environment model [12]. It also enables us to apply easily any other constraints which exist in the market [3].

4 Summary

We use methods borrowed from statistical mechanics to obtain the price density, then retrieve the Bühlmann results on the economic premium principle. It is a completely general formalism and may be expanded to other status markets such as finite size markets. This work is in progress by the author.

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